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1. Learning Outcomes

After studying this module, you shall be able to

- Understand the difference between an ideal classical gas and an ideal quantum gas arising because of indistinguishability of particles leading to quantum effects not observed classically.
- Understand the relevance of mean thermal wavelength or thermal de Broglie wavelength as a deciding parameter to characterize when are the quantum effects significant.
- Understand how the behavior of a Fermi gas is different from a classical ideal gas
- Understand when is fermi gas said to be degenerate
- Learn that since at $T = 0$ fermi distribution has a simple form, and therefore, various physical quantities such as Number Density, Fermi wave vector, Fermi momentum, Fermi Energy and Fermi Temperature of fermi gas at $T = 0$ can be easily calculated
- Learn that at $T \neq 0$, fermi distribution does not remain a step function and around ϵ_F in width approximately $2k_BT$, it changes which influences all the properties and calculations become tricky.
- Calculate thermodynamic properties viz equation of state, number density, internal energy, specific heat, Helmholtz free energy and entropy of a fermi gas at $T \neq 0$ involving fermi integrals.
- Learn that the non-degenerate fermi gas, for which $\frac{N\lambda^3}{Vg} \ll 1$, behaves like a classical ideal gas.
- Apply Sommerfeld's approach to solve fermi integrals and revisit the thermodynamic properties of a fermi gas at finite but low temperature.
- Understand the behavior of an ideal fermi gas in the presence of an external magnetic field where application of fermi-dirac statistics leads to pauli paramanetism and landau's diamagnetism. First one arising from the contribution of alignment of spin of electrons in the direction of the applied magnetic field and the second one arising from the lorentz force experienced by the moving electrons in the presence of a magnetic field producing an induced current opposing the applied magnetic field.
- Calculate Pauli's spin susceptibility at $T = 0$ and at $T \neq 0$.
- Know that how motion of electrons in a magnetic field gives rise to Landau levels and lead to curie like diamagnetic susceptibility.

2. Introduction

In module XV various thermodynamic properties for Fermi-Dirac cases from the knowledge of partition function in Grand Canonical Ensemble were obtained. In this module we shall apply these results to the case of an ideal Fermi gas or non-interacting Fermi gas. The non-interacting nature of the gas allows us to treat each particle independently except for the quantum effect introduced by Pauli's Exclusion Principle. Each particle can be assigned a single particle state with energy specific to that state as we saw in Module XVII. This inherent interaction manifested itself in the form of antisymmetric nature of the wave function. However, it is important to realize that quantum effects become pronounced when the inter-particle distance between the particles is comparable to their average de Broglie Wavelength and wave functions of different particles overlap and system can-not be treated classically. The gas is then said to be degenerate. This happens when fermion gas has very low temperature or very high density.

We derived in earlier modules expectation value of the number of particles in the state

$$\langle \boldsymbol{n_k} \rangle = \frac{\mathbf{1}}{\left(\boldsymbol{e}^{\frac{\epsilon_k - \mu}{k_B T}} + \mathbf{1}\right)} \tag{1}$$

Which when treated as a function of E, gives us the Fermi-Dirac Distribution function

$$\boldsymbol{f}(\boldsymbol{\epsilon_k}) = \frac{\mathbf{1}}{\left(\boldsymbol{e}^{\frac{\epsilon_k - \mu}{k_B T}} + \mathbf{1}\right)} \tag{2}$$

The plot of $\boldsymbol{f}(\boldsymbol{\epsilon_k})$ for different values of $\boldsymbol{T}$ is given in figure 1 below:

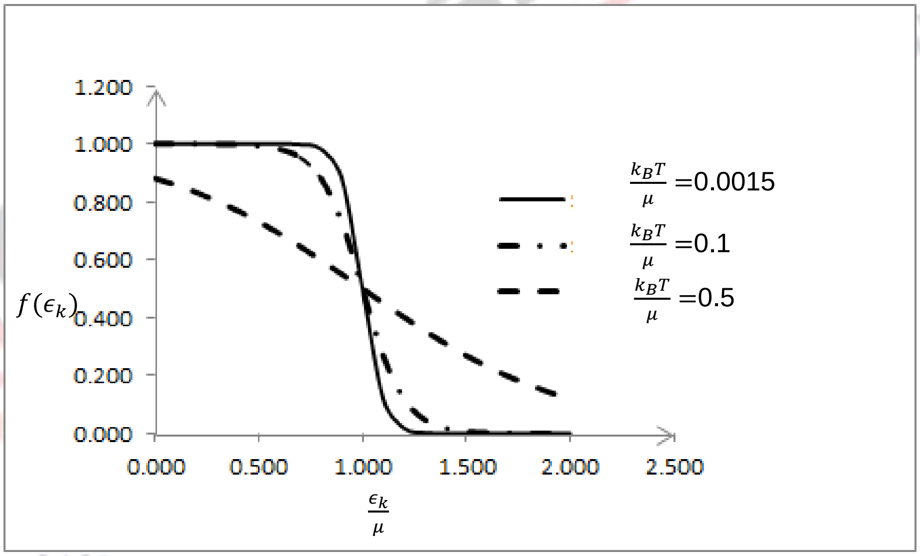


Figure 1 Fermi distribution function for different values of temperature T >0

Fermi distribution function in the limit $\frac{\boldsymbol{\epsilon_k - \mu}}{\boldsymbol{k_B T}} \gg \mathbf{1}$, the equation transforms into $\boldsymbol{e}^{-\frac{\epsilon_k - \mu}{k_B T}}$ which is nothing but Boltzmann distribution function.

The total number of fermi particles is then given by

$$\boldsymbol{N} = \sum_{\boldsymbol{k}} \boldsymbol{f}(\boldsymbol{\epsilon_k}) = \sum_{\boldsymbol{k}} \frac{\mathbf{1}}{\left(\boldsymbol{e}^{\frac{\epsilon_k - \mu}{k_B T}} + \mathbf{1}\right)} \tag{3}$$

At $\boldsymbol{\frac{\mu}{k_B T} \gg 1}$ and $\boldsymbol{\epsilon_k - \mu \ll 0}$, $\boldsymbol{f(\epsilon_k) = 1}$. At $\boldsymbol{\epsilon_k = \mu}$, $\boldsymbol{f(\epsilon_k) = \frac{1}{2}}$. At T=0 Fermi distribution is given in figure 2. And it can be represented as

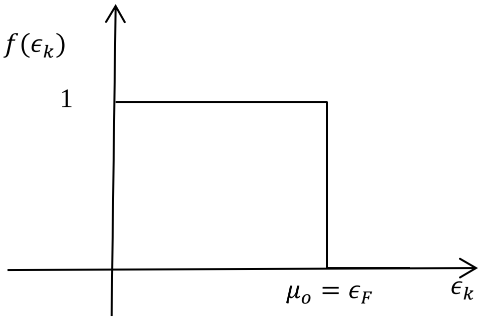


Figure 2 Fermi distribution function at T=0

$$\boldsymbol{f(\epsilon_k) = \begin{cases} 1 \; for \; \epsilon_k < \mu_0 \\ 0 \; for \; \epsilon_k > \mu_0 \end{cases}} \qquad (4)$$

Here $\boldsymbol{\mu_0}$ is chemical potential of the fermi gas at $\boldsymbol{T = 0}$. Most commonly it is known as fermi energy $\boldsymbol{\mu_0 = \epsilon_F}$. Accordingly all the states up to $\boldsymbol{\epsilon_F}$ are completely filled and above it are completely empty. We can, therefore, also define Fermi wave vector $\boldsymbol{k_F}$, fermi momentum $\boldsymbol{p_F = \hbar k_F}$ giving Fermi energy $\boldsymbol{\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m}}$. The summation in equation (3) above can be converted into an integral over $\boldsymbol{k}$, with various energy levels $\boldsymbol{\epsilon_k}$

$$\boldsymbol{\sum_k f(\epsilon_k) = \begin{cases} \frac{V}{(2\pi)^3} \int f(\epsilon_k) d^3k \;\; in \; three \; dimensions \\ \frac{A}{(2\pi)^2} \int f(\epsilon_k) d^2k \;\; in \; two \; dimensions \\ \frac{L}{(2\pi)} \int f(\epsilon_k) \, dk \;\; in \; one \; dimension \\ \frac{L^d}{(2\pi)^d} \int f(\epsilon_k) d^dk \;\; in \;\; d \; dimensions \end{cases}} \qquad (5)$$

Where $\boldsymbol{\frac{L^d}{(2\pi)^d}}$ is the density of states in d dimensions.

If $\boldsymbol{\mathcal{G}}$ denotes the internal degeneracy factor for spin the number of states get multiplied by it. For example in the case of spin of an electron $\boldsymbol{\mathcal{G} = 2}$, one corresponding to spin up (+ ½) and the other corresponding to spin down (-½).

3. Degenerate Fermi Gas

The name degenerate has a special significance here, a system of fermions say electrons is said to degenerate if the average de Broglie wavelength which we encountered earlier in module one is comparable to or much larger than the average particle distance such that their wave functions overlap and that this overlap cannot be ignored and the system can no longer be treated as classical and must be treated quantum mechanically. It is expected that systems at low $\boldsymbol{T}$ that is at high value of $\boldsymbol{\beta}$ will be degenerate. However, a system at high temperatures can be degenerate provided they have high chemical potential. In the case of metals Fermi temperature $\boldsymbol{T_F} = \frac{\boldsymbol{\epsilon_F}}{\boldsymbol{k_B}} \sim \mathbf{4} \times \mathbf{10^5}$K. which is much higher than room temperature and system should be treated quantum mechanically.

4. Ideal Fermi Gas at T=0

At T=0, because of simple form of distribution function, equation (4) calculation of various physical quantities becomes simple. Let us look at the calculation of number density, energy density and pressure of fermi gas at $\boldsymbol{T} = \mathbf{0}$.

(a) **Number density:**

Note

$$\boldsymbol{N} = \sum_{\boldsymbol{k}} \boldsymbol{f(\epsilon_k)} = \frac{\boldsymbol{\mathcal{G}\,V}}{\boldsymbol{(2\pi)^3}} \int \boldsymbol{d^3k} \tag{6}$$

$$\boldsymbol{N} = \frac{\boldsymbol{\mathcal{G}\,4\,\pi V}}{\boldsymbol{(2\pi)^3}} \int_{\mathbf{0}}^{\boldsymbol{k_F}} \boldsymbol{k^2 dk} \tag{7}$$

$$\boldsymbol{N} = \frac{\boldsymbol{\mathcal{G}\,V}}{\boldsymbol{3\,\pi^2}}\,\boldsymbol{k_F^3} \tag{8}$$

$$\frac{\boldsymbol{N}}{\boldsymbol{V}} = \frac{\boldsymbol{\mathcal{G}k_F^3}}{\boldsymbol{3\,\pi^2}} \tag{9}$$

(b) Fermi wave vector, Fermi momentum, Fermi Energy and Pressure

Thus the Fermi wave vector $\boldsymbol{k_F}$ is given by

$$\boldsymbol{k_F} = \left(\frac{\boldsymbol{3\pi^2 N}}{\boldsymbol{\mathcal{G}V}}\right)^{\frac{1}{3}} \tag{10}$$

and Fermi momentum $\boldsymbol{P_F}$ is given by $\boldsymbol{P_F = h\, k_F}$

The energy of the highest occupied k level at $\boldsymbol{T = 0}$ i.e. of $\boldsymbol{k_F}$ is

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{\mathcal{G} V}\right)^{\frac{2}{3}} \tag{11}$$

The total energy, $\boldsymbol{E_o}$ of fermions at $\boldsymbol{T = 0}$ can be calculated by summing kinetic energy over all the states up to Fermi level with $\boldsymbol{k = k_F}$:

$$E_o = \sum_k \frac{\hbar^2 k^2}{2m} = \frac{\mathcal{G}\, 4\,\pi V}{(2\pi)^3}\, \frac{\hbar^2}{2m} \int_0^{k_F} k^4 dk = \frac{\mathcal{G}\, V}{2\,\pi^2}\, \frac{\hbar^2}{2m}\, \frac{k_F^5}{5} \tag{12}$$

Or

$$E_o = \frac{\mathcal{G}\, V}{2\,\pi^2}\, \frac{\hbar^2 k_F^2}{2m}\, \frac{k_F^3}{5} = \frac{\mathcal{G}\, V}{2\,\pi^2}\, \epsilon_F\, \frac{k_F^3}{5} = \frac{\mathcal{G}}{2}\, \frac{3N\epsilon_F}{5} \tag{13}$$

For electrons $\boldsymbol{\mathcal{G} = 2}$ is spin degeneracy, and $\boldsymbol{E_o = \frac{3N\epsilon_F}{5V}}$. It is interesting to note that at T=0, Fermi gas has non-zero kinetic energy and hence the pressure of the gas can be calculated:

$$P = -\frac{\partial E}{\partial V} = -\partial \frac{\frac{\mathcal{G}}{2}\frac{3N}{5}\epsilon_F}{\partial V} = \frac{\mathcal{G}}{2}\frac{2N}{5V}\, \epsilon_F \tag{14}$$

(c) Fermi temperature:

We can define an equivalent temperature corresponding to Fermi energy $\boldsymbol{\epsilon_F}$ as strongly degenerate gas.

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = k_B T_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}} \tag{15}$$

The results derived about Fermi gas at $\boldsymbol{T = 0}$ above are valid at $\boldsymbol{T}$ close to zero. This actually implies whenever $\boldsymbol{T \ll T_F}$. Fermi temperature $\boldsymbol{T_F}$ is also known as degeneracy temperature. This also helps us to classify fermi gas at $\boldsymbol{T = 0}$ as completely degenerate gas and for $\boldsymbol{T \ll T_F}$ as a strongly degenerate gas.

At room temperature (of the order of $\boldsymbol{10^3\ K}$) simple metals such as Sodium can be treated as strongly degenerate Fermi System with $\boldsymbol{T_F}$ being of the order of $\boldsymbol{10^5}$K . It is the reason why quantum mechanical treatment is needed for discussing metals at room temperature.

5. Ideal Fermi gas at T≠ 0

At $\boldsymbol{T \neq 0}$, Figure 1, we cannot approximate Fermi Dirac Distribution function as a step function with value 1 for $\boldsymbol{\epsilon_k < \epsilon_F}$ and 0 for $\boldsymbol{\epsilon_k > \epsilon_F}$. However, at low temperature graph for fermi distribution differs from the graph at $\boldsymbol{T = 0}$ only around $\boldsymbol{\epsilon = \epsilon_F}$ over a width approximately equal to $\boldsymbol{2k_BT}$, a small region around $\boldsymbol{\epsilon_F}$. As T increases this width increases as is clear from figure 1. So the calculations do not remain straight forward as in the case of $\boldsymbol{T = 0}$. In module 15, we wrote expressions for equation of state, for average energy and average number of particles, let us evaluate these to get most fundamental thermodynamic properties of a fermi gas at low temperature $\boldsymbol{T \neq 0}$

5.1 Thermodynamic properties

(a) Equation of State of a Fermi Gas

Recalling from module 15

$$\frac{PV}{k_BT} = \ln \mathbb{Z}(Z, V, T) = \sum_{\epsilon} \ln\left(1 + e^{\frac{\mu-\epsilon}{k_BT}}\right) = \sum_{\epsilon} \ln\left(1 + Ze^{-\beta\epsilon}\right) \tag{16}$$

Where, $\mathbb{Z}$ is grand partition function, $= \frac{1}{k_BT}$, $Z = e^{\frac{\mu}{k_BT}}$ and $\epsilon = \frac{p^2}{2m}$.

On replacing summation over integration equation of state can be obtained as

$$\frac{PV}{k_BT} = \sum_{\epsilon} \ln\left(1 + Ze^{-\beta\epsilon}\right) = \frac{\mathcal{G}V}{(2\pi)^3} \int \ln\left(1 + Ze^{-\beta\epsilon}\right) \, d^3k \tag{17}$$

Where $\mathcal{G}$ is is the degeneracy factor because of the spin.

Or
$$\frac{PV}{k_BT} = \frac{\mathcal{G}V4\pi}{(2\pi)^3} \int_0^\infty \ln\left(1 + Ze^{-\beta\epsilon}\right) \, k^2 \, dk \tag{18}$$

Noting that $\epsilon = \frac{\hbar^2 k^2}{2m}$, $k^2 dk = \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}} d\epsilon$

Or
$$\frac{PV}{k_BT} = \frac{\mathcal{G}V4\pi}{2(2\pi)^3} \int_0^\infty \ln\left(1 + Ze^{-\beta\epsilon}\right) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}} d\epsilon \tag{19}$$

Put $\beta\epsilon = x$, equation (19) can be written as

Or
$$\frac{P}{k_BT} = \mathcal{G}\left(\frac{2\,\pi m k_B T}{h^2}\right)^{\frac{3}{2}} \left(\frac{4}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \ln\left(1 + Ze^{-x}\right) \; x^{\frac{1}{2}} dx \tag{20}$$

or
$$\frac{P}{k_BT} = \frac{\mathcal{G}}{\lambda^3}\left(\frac{4}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \ln\left(1 + Ze^{-x}\right) \; x^{\frac{1}{2}} dx \tag{21}$$

Where $\frac{1}{\lambda^3} = \left(\frac{2\,\pi m k_B T}{h^2}\right)^{\frac{3}{2}}$, and $\lambda = \frac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$ is thermal de Broglie wavelength.

Let us see how $I = \int_0^\infty \ln(1+Ze^{-x})\ x^{\frac{1}{2}} dx$ can be evaluated, it can be integrated by parts

$$I = \ln(1+Ze^{-x})x^{3/2}\Big|_0^\infty + \frac{2}{3}\int_0^\infty \frac{Ze^{-x}}{(1+Ze^{-x})} x^{\frac{3}{2}} dx \tag{22}$$

Here the first term vanishes, since first term of the product vanishes at upper limit and second term of the limit vanishes at lower limit.

Therefore, equation (22) reduces to

or

$$I = \frac{2}{3}\int_0^\infty \frac{1}{(Z^{-1}e^x+1)} x^{\frac{3}{2}} dx \tag{23}$$

or

$$\frac{P}{k_B T} = \frac{\mathcal{G}}{\lambda^3}\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \frac{4}{3}\int_0^\infty \frac{x^{\frac{3}{2}}}{(Z^{-1}e^x+1)} dx = \frac{\mathcal{G}}{\lambda^3} f_{\frac{5}{2}}(z) \tag{24}$$

Where $f_{\frac{5}{2}}(z) = \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^\infty \frac{x^{\frac{3}{2}}}{(Z^{-1}e^x+1)} dx$, which is a generic integral of the type

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{(Z^{-1}e^x+1)} dx \tag{25}$$

These integrals are known as Fermi-Dirac integrals (See Appendix-A2). And for small z, this integral can be expanded as a convergent series such that

$$f_\nu(z) = z - \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} - \cdots \tag{26}$$

(b) Number density

Recall

$$N = \sum_\epsilon < n_\epsilon > = \sum_\epsilon \frac{1}{Z^{-1}e^{\beta\epsilon}+1} \tag{27}$$

Summation can be converted into an integral as

$$N = \frac{V\mathcal{G}}{\lambda^3}\left(\frac{4}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{1}{Z^{-1}e^{\beta\epsilon}+1}\; x^{\frac{1}{2}}dx = \frac{V\mathcal{G}}{\lambda^3}\, f_{\frac{3}{2}}(Z) \tag{28}$$

or

$$\frac{N}{V} = \frac{\mathcal{G}}{\lambda^3}\, f_{\frac{3}{2}}(Z) \tag{29}$$

Using equation (24) and (29) we can get the *equation of state* of the fermi gas as

$$\frac{PV}{Nk_BT} = \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(Z)} \tag{30}$$

(c) Internal Energy

Recalling from module XV that internal energy of a fermionic gas can be written as

$$U \equiv -\left(\frac{\partial}{\partial\beta}\ln \mathbb{Z}(Z,V,T)\right) = k_BT^2\left(\frac{\partial}{\partial T}\ln \mathbb{Z}(Z,V,T)\right)_{z,V} \tag{31}$$

Or

$$U = \mathcal{G}Vf_{\frac{5}{2}}(z)k_BT^2\left(\frac{\partial}{\partial T}\,\frac{1}{\lambda^3}\right)_{z,V} = \mathcal{G}Vf_{\frac{5}{2}}(z)\left(\frac{2\,\pi m k_B}{h^2}\right)^{\frac{3}{2}}k_BT^2\,\frac{\partial\,(T)^{\frac{3}{2}}}{\partial T} \tag{32}$$

$$= \frac{3}{2}k_BT\left(\frac{\mathcal{G}V}{\lambda^3}\right)f_{\frac{5}{2}}(z) = \frac{3}{2}Nk_BT\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}$$

From (30) it follows that

$$U = \frac{3}{2}Nk_BT\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} = \frac{3}{2}PV \tag{33}$$

And a general relation connecting temperature with energy density emerges in the case of a Fermi gas

$$P = \frac{2}{3}\left(\frac{U}{V}\right) \tag{34}$$

A formula which holds for ideal classical gas as well.

(d) Specific heat

By differentiating internal energy $\boldsymbol{U}$ (equation (32)) with respect to $\boldsymbol{T}$ we can get the specific heat $\boldsymbol{C_V}$ as

$$\frac{C_V}{NK_B} = \left(\frac{\partial}{\partial T}\left(\frac{3}{2}T\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\right)\right) = \frac{3}{2}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + \frac{3}{2}T\frac{\partial}{\partial T}\left(\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}\right) \tag{35}$$

$$= \frac{3}{2}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + T\frac{1}{Z}\frac{\left(f_{\frac{5}{2}}(z)z\frac{\partial f_{\frac{3}{2}}(z)}{\partial z}\frac{\partial z}{\partial T} - f_{\frac{3}{2}}(z)z\frac{\partial f_{\frac{5}{2}}(z)}{\partial z}\frac{\partial z}{\partial T}\right)}{\left(f_{\frac{3}{2}}(z)\right)^2}$$

Recalling that

$$Z\frac{\partial f_\nu(z)}{\partial z} = \frac{\partial f_\nu(z)}{\partial\,(\ln z)} = f_{\nu-1}(z)$$

Equation (35) can be written as

$$\frac{C_V}{NK_B} = \frac{3}{2}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + \frac{3}{2}\frac{1}{Z}\frac{\partial z}{\partial T}\frac{\left(f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z) - f_{\frac{3}{2}}(z)f_{\frac{3}{2}}(z)\right)}{\left(f_{\frac{3}{2}}(z)\right)^2}$$

$$= \frac{3}{2}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{3}{2}T\frac{1}{Z}\frac{\partial z}{\partial T}\left(1 - \frac{\left(f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)\right)}{\left(f_{\frac{3}{2}}(z)\right)^2}\right) \tag{36}$$

Now to get $\frac{\partial z}{\partial T}$, let us recall equation (29)

$f_{\frac{3}{2}}(Z) = \frac{N\lambda^3}{V\mathcal{G}}$ Then

$$Z\frac{\partial f_{\frac{3}{2}}(Z)}{\partial Z}\frac{1}{Z}\frac{\partial Z}{\partial T} = \frac{N}{V\mathcal{G}}\frac{\partial\lambda^3}{\partial T} = -\frac{3}{2T}\frac{N}{V\mathcal{G}}\lambda^3 \tag{37}$$

Or $$f_{\frac{1}{2}}(Z)\frac{1}{Z}\frac{\partial Z}{\partial T} = \frac{N}{V\mathcal{G}}\frac{\partial\lambda^3}{\partial T} = \frac{N}{V\mathcal{G}}\left(-\frac{3}{2T}\lambda^3\right) = -\frac{3}{2T}\frac{N}{V\mathcal{G}}\lambda^3 = -\frac{3}{2T}f_{\frac{3}{2}}(Z) \tag{38}$$

Or
$$\frac{1}{Z}\frac{\partial Z}{\partial T} = -\frac{3}{2T}\frac{f_{\frac{3}{2}}(Z)}{f_{\frac{1}{2}}(Z)} \tag{39}$$

Therefore, equation (36) becomes

$$\begin{aligned}\frac{C_V}{NK_B} &= \frac{3}{2}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} + \frac{3}{2}\frac{1}{Z}\frac{\partial z}{\partial T}\frac{\left(f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z) \;-\; f_{\frac{3}{2}}(z)f_{\frac{3}{2}}(z)\right)}{\left(f_{\frac{3}{2}}(z)\right)^2} \\ &= \frac{3}{2}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{9}{4}\frac{f_{\frac{3}{2}}(Z)}{f_{\frac{1}{2}}(Z)}\left(1 - \frac{\left(f_{\frac{5}{2}}(z)f_{\frac{1}{2}}(z)\right)}{\left(f_{\frac{3}{2}}(z)\right)^2}\right) \\ &= \frac{3}{2}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{9}{4}\left(\frac{f_{\frac{3}{2}}(Z)}{f_{\frac{1}{2}}(Z)} - \frac{\left(f_{\frac{5}{2}}(z)\right)}{\left(f_{\frac{3}{2}}(z)\right)}\right) \\ &= \frac{15}{4}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{9}{4}\left(\frac{f_{\frac{3}{2}}(Z)}{f_{\frac{1}{2}}(Z)}\right)\end{aligned} \tag{40}$$

(e) Helmholtz free energy:

Recall that Helmholtz free energy of is $A \equiv N\mu - PV$, for the ideal fermi gas it can obtained using equation (30) as

$$A = Nk_BT\left(\ln Z - \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(Z)}\right) \tag{41}$$

(f) Entropy

Recall entropy is given by $S \equiv \frac{(U-A)}{T}$, so entropy for the fermi gascan be written as

$$S = \left(\frac{5}{2}Nk_BT\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \ln Z\right) \tag{42}$$

The relationships derived above give us means to determine properties of the ideal fermi gas in terms of number density $\left(\frac{N}{V}\right)$, and temperature T provided we know the functional dependence of parameter Z on number density and temperature T. This dependence is known to us via the equation (29). Though the function $f_\nu(Z)$ involved can be calculated numerically in the following we will confine ourselves to various limiting approximations of the functions involved.

5.2 Non-Degenerate Fermi Gas $\left(\frac{N\lambda^3}{V\mathcal{G}} \ll 1\right)$

This corresponds to a case when $f_{\frac{3}{2}}(Z) = \frac{N\lambda^3}{V\mathcal{G}} \ll 1$ i.e. density of the fermi gas is very low and its temperature is very high or number density may not be low but temperature very high. In this case gas is said to be non-degenerate and behaves like a classical ideal gas. Recalling series expansion of $f_\nu(z)$ in equation (26) this implies that $f_\nu(z) \cong Z$ and thermodynamic properties become

$$\frac{PV}{Nk_BT} = \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(Z)} = \frac{Z}{Z} = 1 \; i.e. PV = Nk_BT \qquad (43)$$

$$\frac{C_V}{NK_B} = \frac{15}{4}\frac{Z}{Z} - \frac{9}{4}\left(\frac{Z}{Z}\right) = \frac{3}{2} \text{ i.e. } C_V = \frac{3}{2} Nk_B \qquad (44)$$

$$\begin{aligned} A &= Nk_BT\left(\ln Z - \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(Z)}\right) = Nk_BT(\ln Z - 1) \\ &= Nk_BT\left(\ln\left(\frac{N\lambda^3}{V\mathcal{G}}\right) - 1\right) \end{aligned} \qquad (45)$$

$$S = \left(\frac{5}{2}Nk_BT\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \ln Z\right) = \left(\frac{5}{2}Nk_BT - \ln\left(\frac{N\lambda^3}{V\mathcal{G}}\right)\right) \qquad (46)$$

5.3 Fermi Gas at Finite But Low Temperature (Sommerfeld's approach):

For discussing this we shall follow Sommerfeld's approach, reminding ourselves that as temperature rises above 0^oK to a low but finite value, the thermal excitation of the particles occur in a narrow range about $\epsilon = \epsilon_F$, called fermi energy.

To proceed further, let us revisit two results, equation (24) and (28) we obtained earlier in deriving equation of state, number density. These two will pave way for calculating internal energy, Pressure, specific heat, entropy and Helmholtz free energy.

(a) Equation of State & Number Density

$$\frac{PV}{k_BT} = \frac{\mathcal{G}V}{\lambda^3}\left(\frac{1}{\pi}\right)^{\frac{1}{2}}\frac{4}{3}\int_0^\infty \frac{(\beta\epsilon)^{\frac{3}{2}}}{(\,e^{\beta(\epsilon-\mu)}+1)}d(\beta\epsilon) = \frac{\mathcal{G}V}{\lambda^3}\;\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^\infty \frac{(\beta\epsilon)^{\frac{3}{2}}}{(e^{\beta(\epsilon-\mu)}+1)}d(\beta\epsilon)$$

Or

$$= \frac{\mathcal{G}V}{\lambda^3}\;\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_{-\alpha}^\infty \frac{(\xi+\alpha)^{\frac{3}{2}}}{(e^{\xi}+1)}d\xi$$

$$N = \frac{V\mathcal{G}}{\lambda^3}\left(\frac{4}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{1}{Z^{-1}e^{\beta\epsilon}+1}\;(\beta\epsilon)^{\frac{1}{2}}dx = \frac{V\mathcal{G}}{\lambda^3}\;\frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_0^\infty \frac{(\beta\epsilon)^{\frac{1}{2}}}{(e^{\beta(\epsilon-\mu)}+1)}d(\beta\epsilon)$$

$$= \frac{V\mathcal{G}}{\lambda^3}\;\frac{1}{\Gamma\left(\frac{3}{2}\right)}\int_{-\alpha}^\infty \frac{(\xi+\alpha)^{\frac{1}{2}}}{(e^{\xi}+1)}d\xi$$

After putting $\beta(\epsilon-\mu)=\xi$ and $\beta\mu=\alpha$ the two integrals so called fermi integrals can be cast in a generic form and solved

$$\mathcal{F}(\alpha) = \int_{-\alpha}^{\infty}\frac{\mathbb{G}(\xi+\alpha)}{e^{\xi}+1}\,d\xi$$

Where $\mathbb{G}(\xi+\alpha)$ is a polynomial.

***Sommerfeld's approach to solve Fermi integrals**:*

To solve these types integrals an approach was suggested by Sommerfeld to study properties of metals at finite temperature and is as follows:

This generic integral can can be written in the form

$$\mathcal{F}(\alpha) = \int_{-\alpha}^{0}\frac{\mathbb{G}(\xi+\alpha)}{e^{\xi}+1}\,d\xi + \int_{0}^{\infty}\frac{\mathbb{G}(\xi+\alpha)}{e^{\xi}+1}\,d\xi \qquad (47)$$

Noting that

$$\frac{1}{e^{\xi}+1} = 1 - \frac{1}{e^{-\xi}+1}$$

(47) can be written as

$$
\begin{aligned}
\mathcal{F}(\alpha) &= \int_{-\alpha}^{0} \mathbb{G}(\xi+\alpha)\, d\xi + \int_{-\alpha}^{0} \frac{\mathbb{G}(\xi+\alpha)}{e^{-\xi}+1}\, d\xi + \int_{0}^{\infty} \frac{\mathbb{G}(\xi+\alpha)}{e^{\xi}+1}\, d\xi \\
&= \int_{0}^{\alpha} \mathbb{G}(\xi)\, d\xi + \int_{0}^{\alpha} \frac{\mathbb{G}(\alpha-\xi)}{e^{\xi}+1}\, d\xi + \int_{0}^{\infty} \frac{\mathbb{G}(\xi+\alpha)}{e^{\xi}+1}\, d\xi \\
&= \int_{0}^{\alpha} \mathbb{G}(\xi)\, d\xi + \int_{0}^{\infty} \frac{\mathbb{G}(\xi+\alpha)-\mathbb{G}(\alpha-\xi)}{e^{\xi}+1}\, d\xi - \int_{\alpha}^{\infty} \frac{\mathbb{G}(\alpha-\xi)}{e^{\xi}+1}\, d\xi
\end{aligned} \qquad (48)
$$

Again noting that $\alpha \gg 1$, the last term is of the order of $e^{-\alpha}$ and can be neglected. Numerator in the integrand of the second term can be expanded around α to get

$$
\mathbb{G}(\xi+\alpha) - \mathbb{G}(\alpha-\xi) = 2\xi\, \mathbb{G}'^{(\alpha)} + \frac{\xi^3}{3} \mathbb{G}'''^{(\alpha)} \ldots
$$

Therefore (47) can be written as

$$
\mathcal{F}(\alpha) = \int_{0}^{\alpha} \mathbb{G}(\xi)\, d\xi + 2\, \mathbb{G}'^{(\alpha)} \int_{0}^{\infty} \frac{\xi}{e^{\xi}+1}\, d\xi + \frac{\mathbb{G}'''(\alpha)}{3} \int_{0}^{\infty} \frac{\xi^3}{e^{\xi}+1}\, d\xi + \cdots \qquad (49)
$$

(49) involves integrals of the form $\int_0^\infty \frac{\xi^n}{e^{\xi}+1}\, d\xi$, which can be evaluated as follows

$$
\begin{aligned}
\int_0^\infty \frac{Z^{x-1}}{e^Z+1}\, dZ &= \int_0^\infty e^{-z} Z^{x-1}\, (e^{-z}+1)^{-1}\, d\xi \\
&= \int_0^\infty e^{-Z} Z^{x-1} \sum_{n=0}^{\infty} (-1)^n\, e^{-nZ}\, dZ \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-(n+1)z}\, Z^{x-1}\, dz \\
&= \Gamma(x) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x} = (1-2^{1-x})\Gamma(x) \sum_{n=1}^{\infty} \frac{1}{n^x} \\
&= (1-2^{1-x})\Gamma(x)\zeta(x)
\end{aligned} \qquad (50)
$$

Where $\Gamma(x)$ is gamma function and $\zeta(x)$ is Riemann-Zeta function for $x > 0$.

Therefore,

$$
\int_0^\infty \frac{\xi}{e^{\xi}+1}\, d\xi = \left(1-2^{1-2}\right)\Gamma(2)\zeta(2) = \frac{1}{2}.1.\frac{\pi^2}{6} = \frac{\pi^2}{12}
$$

And

$$
\int_0^\infty \frac{\xi^3}{e^{\xi}+1}\, d\xi = \left(1-2^{1-4}\right)\Gamma(4)\zeta(4) = \frac{7}{8}.6.\frac{\pi^4}{90} = \frac{7\pi^4}{120}
$$

So that (49) may be written as

$$\mathcal{F}(\alpha) = \int_0^{\alpha} \mathbb{G}(\xi)\, d\xi + \frac{\pi^2}{6}\, \mathbb{G}'(\alpha) + \frac{7\,\pi^4}{360}\, \mathbb{G}'''(\alpha) + \cdots \qquad (51)$$

So when $\mathbb{G}(\xi) = \xi^{\frac{1}{2}}$

$$\mathcal{F}(\alpha) = \frac{2}{3}\, \alpha^{\frac{3}{2}} + \frac{\pi^2}{12}\, \alpha^{-\frac{1}{2}} + \frac{7\,\pi^4}{360} \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \alpha^{-\frac{5}{2}} + \cdots \qquad (52)$$

And when $\mathbb{G}(\xi) = \xi^{\frac{3}{2}}$

$$\mathcal{F}(\alpha) = \frac{2}{5}\, \alpha^{\frac{5}{2}} + \frac{3\pi^2}{12}\, \alpha^{\frac{1}{2}} + \frac{7\,\pi^4}{360} \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \alpha^{-\frac{3}{2}} + \cdots \qquad (53)$$

(b) Internal Energy

So the internal energy can be calculated using (53)

$$\begin{aligned} U = \frac{3}{2} PV &= \frac{3}{2}\frac{\mathcal{G}V}{\lambda^3}\; k_B T\, \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_{-\alpha}^{\infty} \frac{(\xi+\alpha)^{\frac{3}{2}}}{(e^{\xi}+1)} d\xi \\ &= \frac{4\pi\mathcal{G}V}{5} \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \mu^{\frac{5}{2}} \left[1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2 + \textit{higher order terms}\right] \end{aligned} \qquad (54)$$

(c) Number Density and Fermi Energy

Using (52) we can calculate number density $\frac{N}{V}$

$$\frac{N}{V} = \frac{\mathcal{G}}{\lambda^3} \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{-\alpha}^{\infty} \frac{(\xi+\alpha)^{\frac{1}{2}}}{(e^{\xi}+1)} d\xi = \frac{4\pi\mathcal{G}}{3} \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} (\mu)^{\frac{3}{2}} \left[1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2 + \textit{higher order terms}\right] \qquad (55)$$

At T=0, we get $\boldsymbol{\mu = \mu_0 = \epsilon_F}$

$$\mu_0 = \epsilon_F = \left(\frac{3N}{4\pi\mathcal{G}V}\right)^{\frac{2}{3}} \frac{h^2}{2m} \qquad (56)$$

Therefore, (55) becomes

$$\begin{aligned} \mu &= \left(\frac{3N}{4\pi\mathcal{G}V}\right)^{\frac{2}{3}} \frac{h^2}{2m} \left[1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2 + \textit{higher order terms}\right]^{-\frac{2}{3}} \\ &\cong \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \textit{higher order terms}\right] \end{aligned} \qquad (57)$$

A result which matches with equation (11) calculated earlier. Figure 3, shows plot of $\frac{\mu}{\epsilon_F}$ vs $\frac{T}{T_F}$

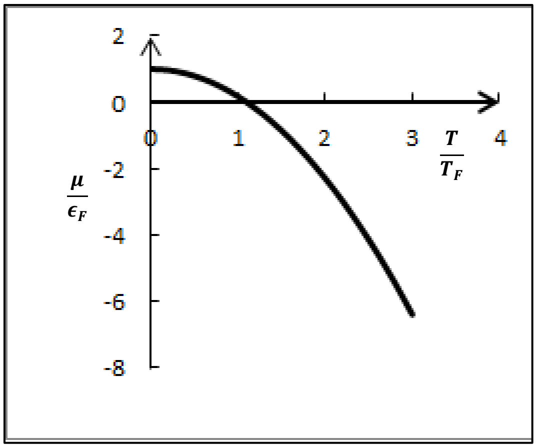


Figure 3 Temperature dependence of chemical potential of Fermi gas $\frac{\mu}{\epsilon_F}$ vs $\frac{T}{T_F}$

Using (56) and (57), and noting that $\frac{4\pi GV}{3N}\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} = (\epsilon_F)^{-3/2}$ we can get from (53)

$$\frac{U}{N} = \frac{3}{5}(\epsilon_F)^{-3/2}(\epsilon_F)^{\frac{5}{2}}\left[1 - \frac{\pi^2}{12}\left(\frac{k_BT}{\mu}\right)^2 + \textit{higher order terms}\right]^{5/2}\left[1 + \frac{5\pi^2}{8}\left(\frac{k_BT}{\mu}\right)^2 + \textit{higher order terms}\right] \tag{58}$$

Or
$$\frac{U}{N} \cong \frac{3}{5}\epsilon_F\left[1 - \frac{5\pi^2}{24}\left(\frac{k_BT}{\mu}\right)^2 + \frac{5\pi^2}{8}\left(\frac{k_BT}{\mu}\right)^2 + \textit{higher order terms}\right] \tag{59}$$

Or
$$\frac{U}{N} \cong \frac{3}{5}\epsilon_F\left[1 + \frac{5\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2 + \textit{higher order terms}\right] \tag{60}$$

(d) Pressure

We can also get pressure P of fermi gas from (59),

$$P = \frac{2}{3}\frac{U}{V} = \frac{2}{5}\frac{N}{V}\epsilon_F\left[1 + \frac{5\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2 + \textit{higher order terms}\right] \tag{61}$$

Or
$$\frac{PV}{Nk_BT_F} = \frac{2}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \textit{higher order terms}\right] \tag{62}$$

Temperature dependence of $\frac{PV}{Nk_BT_F}$ vs $\frac{T}{T_F}$ is shown in figure 4 below

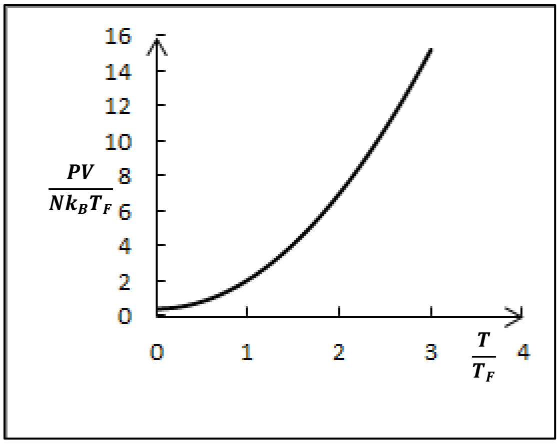


Figure 4 Temperature dependence of $\frac{PV}{Nk_BT_F}$ vs $\frac{T}{T_F}$ of Fermi gas $\frac{\mu}{\epsilon_F}$ vs $\frac{T}{T_F}$

(e) Low temperature specific heat

Using (60), we can get low temperature specific heat at constant volume

Or $$\frac{C_V}{Nk_B} = \frac{\pi^2}{2}\frac{k_BT}{\epsilon_F} + \cdots = \frac{\pi^2}{2}\frac{T}{T_F} + \cdots \quad (63)$$

Where $T_F = \frac{\epsilon_F}{k_B}$ is called fermi temperature. Equation (63) shows that at low temperature specific heat varies linearly with temperature, Figure 5, and is much smaller than the classical value of an ideal gas 3/2 .

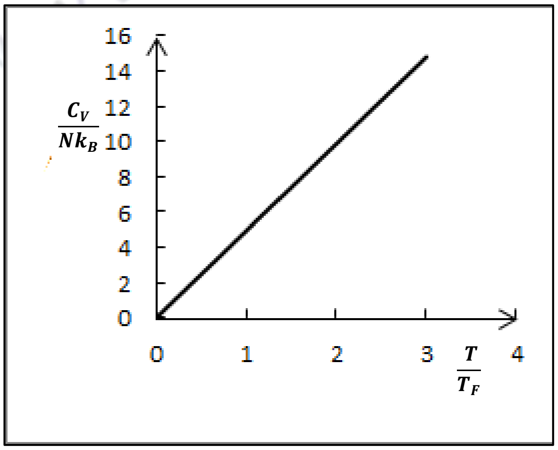


Figure 5 Temperature dependence of specific heat of Fermi gas $\frac{C_V}{Nk_B}$ vs $\frac{T}{T_F}$

(f) Helmholtz Free Energy

Using equations (57) and (58) we can immediately calculate Helmholtz free energy

$$\boldsymbol{F = \mu N - PV} \tag{64}$$

$$\boldsymbol{F = \frac{3}{5} N\epsilon_F \left[1 - \frac{5\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + higher\ order\ terms\right]} \tag{65}$$

(g) Entropy

From (65) we can get entropy using $\boldsymbol{S = -\left(\frac{\partial F}{\partial T}\right)}$ as

$$\boldsymbol{S = Nk_B\left[\frac{\pi^2}{2}\frac{k_B T}{\epsilon_F} + higher\ order\ terms\right]} \tag{66}$$

Here one can see that third law of thermodynamics follows, since as $\boldsymbol{T \to 0\ S \to 0}$

6. Ideal Fermi Gas in a Magnetic Field

As applications of study of ideal fermi systems we now turn our attention to the study of an ideal (non-interacting) Fermi gas at equilibrium in the presence of an external magnetic field.

The practical realization of a fermion gas is electrons in metals and alkali metals are the best candidates in which fermion gas can be treated as a free degenerate electron gas. These electrons are moving, have a mass m, intrinsic spin ½ and a magnetic moment $\boldsymbol{\mu_B = \frac{e\hbar}{2mc}}$ called Bohr magneton and a momentum $\boldsymbol{p = \hbar k}$. On application of external magnetic field there are two possibilities:

i. The magnetic moment of each electron gets aligned in the direction of the magnetic field which results in para-magnetism known as ***Pauli's paramagnetism***. Pauli was a student of Sommerfeld, and Sommerfeld had a lot of interest in improving the understanding of free electron theory of metals proposed earlier by Drude using Boltzmann statistics, He suggested Pauli to study this problem by applying new Fermi-Dirac statistics and Pauli became the first person to study the contribution of spin alignment to the magnetic moment of a degenerate electron gas .
ii. Furthermore, since electrons are moving in the presence of magnetic field they follow a helical path under the influence of Lorentz force, such that current produced by them induces a magnetic field in a direction opposite to external magnetic field resulting in diamagnetism. This was studied first of all by Russian Physicist Lev Landau and is known after his name as ***Landau diamagnetism***.

Here we propose to discuss these two effects in the context of a degenerate fermi gas of electrons.

6.1 Pauli Paramagnetism

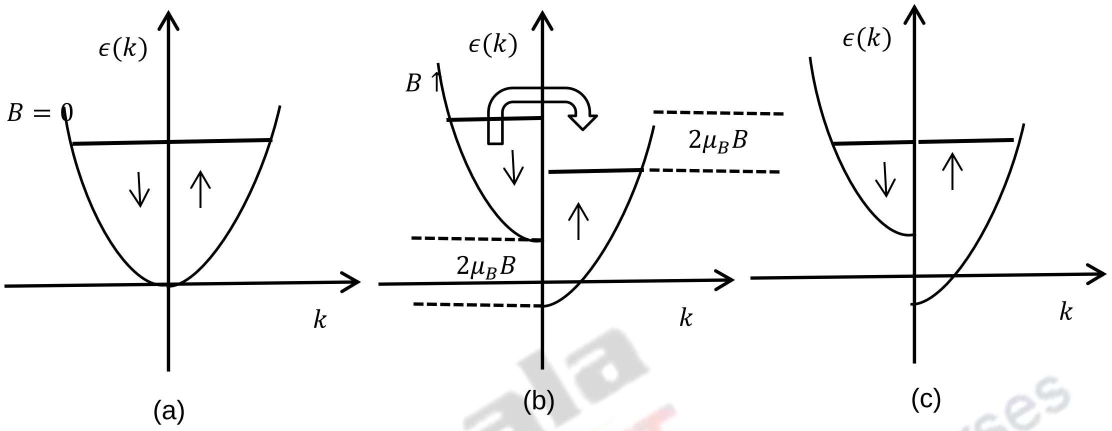


Figure 6 (a) Free electrons without external magnetic field, $\boldsymbol{N_+ = N_- = \frac{N}{2}}$; (b) In the presence of magnetic field $\boldsymbol{B}$ $\boldsymbol{N_-}$ electrons move up by energy $\boldsymbol{\mu_B B}$, and $\boldsymbol{N_+}$ electrons move down by energy $\boldsymbol{\mu_B B}$; (c) To bring the chemical potential same electrons in the state ↑ move to state ↓, such that $\boldsymbol{N_+ > N_-}$ resulting in net magnetization

Let us take a total of N electrons confined a volume of a metal. In the absence of magnetic field the energy levels are filled up to Fermi level $\boldsymbol{\epsilon_F}$, with as many number of electrons with up spin as with down with no net magnetization, Figure 6(a). When magnet field is applied $\boldsymbol{N_+}$ number of electrons get aligned parallel to $\vec{\boldsymbol{B}}$ and $\boldsymbol{N_-}$ number of electrons get aligned anti-parallel to $\vec{\boldsymbol{B}}$, Figure 6(b). Such that $\boldsymbol{N = N_+ + N_-}$. The transfer of electrons from up spin state to down spin state takes place till the chemical potential is the same, Figure 6(c).

An electron with spin up in the presence of magnetic field then has energy

$$\boldsymbol{\epsilon_+ = \frac{p^2}{2m} - \mu_B B} \tag{67}$$

and energy of an electron with spin down in the presence of a magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{\epsilon_- = \frac{p^2}{2m} + \mu_B B} \tag{68}$$

If $\boldsymbol{N_+ > N_-}$, there is net magnetization such that

$$\boldsymbol{M = \mu_B(N_+ - N_-)} \tag{69}$$

We define spin polarization r as

$$r = \frac{N_+ - N_-}{N_+ + N_-} \tag{70}$$

Such that

$$N_+ = \frac{N}{2}(1 + r) \tag{71}$$

And

$$N_- = \frac{N}{2}(1 - r) \tag{72}$$

(a) Case I: T=0

At absolute zero all energy levels up to $\epsilon_F$are filled and above $\epsilon_F$are empty. So kinetic energy of the electrons with spin parallel to B will lie between 0 and $\epsilon_F + \mu_B$ and the electrons with spin antiparallel to B will lie between 0 and $\epsilon_F - \mu_B$.

The number of electrons N_+ with spin parallel and N_- with spin antiparallel to B can be obtained respectively as

$$N_+ = \frac{4\pi V}{3h^3}\left(2m(\epsilon_F + \mu_B B)\right)^{\frac{3}{2}} \tag{73}$$

And

$$N_- = \frac{4\pi V}{3h^3}\left(2m(\epsilon_F - \mu_B B)\right)^{\frac{3}{2}} \tag{74}$$

The net magnetic moment of the fermi electron gas can then be written as

$$M = (N_+ - N_-)\mu_B = \frac{4\pi V(2m)^{\frac{3}{2}}}{3h^3}\left((\epsilon_F + \mu_B B)^{\frac{3}{2}} - (\epsilon_F - \mu_B B)^{\frac{3}{2}}\right)\mu_B \tag{75}$$

In the low limit case, ie. $B \to 0$, we can write it as

$$M = (N_+ - N_-)\mu_B = \frac{4\pi V(2m)^{\frac{3}{2}}}{3h^3}(\epsilon_F)^{\frac{1}{2}} 3\mu_B^2 B \tag{76}$$

So that magnetic field susceptibility per unit volume χ_0, can be written as

$$\chi_0 = M/VB = \frac{4\pi\mu_B^2\,(2m)^{\frac{3}{2}}}{h^3}(\epsilon_F)^{\frac{1}{2}} \qquad (77)$$

Recalling that $\epsilon_F = \left(\frac{3N}{4\pi\mathcal{G}V}\right)^{\frac{2}{3}}\frac{h^2}{2m}$ and $\mathcal{G} = 2$, we have

$$\chi_0 = M/VB = (N_+ - N_-)\mu_B = \frac{\mu_B^2}{(\epsilon_F)}\frac{3N}{2V} = \frac{3}{2}\frac{n\,\mu_B^2}{(\epsilon_F)} \qquad (78)$$

At T=0, the highest occupied energy level of group of up spin electrons $\{N_+\}$is at

$$\epsilon_{F+} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 N_+}{V}\right)^{\frac{2}{3}} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}}(1+r)^{\frac{2}{3}} = \frac{\hbar^2}{2m}k_F^2(1+r)^{2/3} \qquad (79)$$
$$= \epsilon_F(1+r)^{\frac{2}{3}}$$

Similarly, at T=0, the lowest occupied energy level of group of down spin electrons $\{N_-\}$ is at

$$\epsilon_{F-} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 N_-}{V}\right)^{\frac{2}{3}} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}}(1-r)^{\frac{2}{3}} = \frac{\hbar^2}{2m}k_F^2(1-r)^{2/3} \qquad (80)$$
$$= \epsilon_F(1+r)^{\frac{2}{3}}$$

Case II: At Finite Temperature

At finite temperature, ϵ_F needs to taken as $\epsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\right)$,

Therefore. The energy difference between highest up spin energy level and down spin energy levels is given by, Figure 6(b)

$$\epsilon_F[\{(1+r)^{\frac{2}{3}} - (1-r)^{\frac{2}{3}}\} - \frac{\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\{(1+r)^{-\frac{2}{3}} - (1-r)^{-\frac{2}{3}}\}] = 2\mu_B B \qquad (81)$$

For small value of r<<1, equation (81) can be written as

$$\frac{4}{3}r\,\epsilon_F\left[1 + \frac{\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\right] = 2\mu_B B \qquad (82)$$

Such that, spin polarization r is

$$r \approx \frac{3}{2}\frac{\mu_B B}{\epsilon_F}\left[1 + \frac{\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\right] \qquad (83)$$

Then magnetization can be written as

$$M \approx \mu_B r N = \frac{3}{2}\frac{\mu_B^2 N B}{\epsilon_F}\left[1+\frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right] \tag{84}$$

Thus Pauli susceptibility is given by

$$\chi \approx \frac{M}{BV} = \frac{3}{2}\frac{\mu_B^2 N}{\epsilon_F V}\left[1+\frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right] = \chi_0\left[1+\frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right] \tag{85}$$

6.2 Landau Diamagnetism

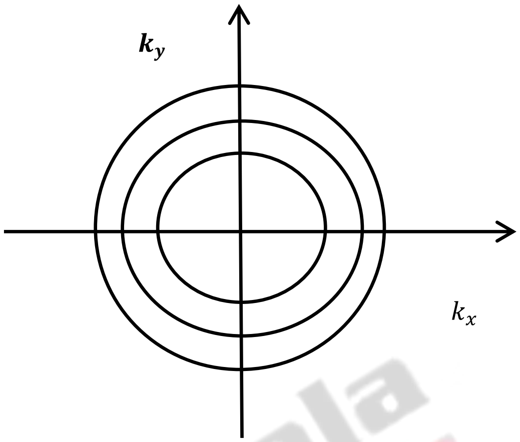
Let us look at motion of a charged particle in a uniform magnetic field $\boldsymbol{B}$ applied in the direction of Z-axis. Such a charged particle follows a helical path with its axis in the direction of Z-axis amounting to linear motion in the direction of Z-axis and circular motion with angular frequency $\frac{eB}{mC}$ in the x-y plane which can be visualized as a simple harmonic motion quantized in the units of $\frac{eB}{mC}\hbar$. The linear energy in the direction of z-axis is also quantized but energy intervals being very small it can be treated as continuous.

For electrons in a metal, therefore the energy levels in a magnetic field can be written as .

$$\epsilon = \frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right)+\frac{\hbar^2 k_z^2}{2m} \tag{86}$$

With n as an index of the each energy level. This means that energy levels or orbits in the $\boldsymbol{k_x}, \boldsymbol{k_y}$ plane gets bunched into a set of allowed energy levels, called landau levels, Figure 7, with the radius square in the $\boldsymbol{k_x}, \boldsymbol{k_y}$ plane given by

$$\boldsymbol{k_x^2 + k_y^2 = \frac{e\hbar B}{mc}\left(n + \frac{1}{2}\right)}$$



(87)

Figure 1 Available landau orbits for a system of charged particles in the presence of a magnetic field in the direction of Z-axis

Let us now focus on the calculation of degeneracy of the nth level is the total number of levels

$$\boldsymbol{\mathcal{G} = \frac{1}{\hbar^2}\frac{1}{(2\pi)^2}\int dx\, dy\, dp_x\, dp_y}$$

$$\boldsymbol{= \frac{1}{\hbar^2(2\pi)^2}\int dx\, dy\, dk_x dk_y = \frac{1}{(2\pi)^2}L_x L_y.\, 2\pi\frac{\hbar^2 k^2}{2}} \tag{88}$$

Or $$\boldsymbol{\mathcal{G} = \frac{2m}{\hbar^2(2\pi)^2}L_x L_y.\, 2\pi\frac{\hbar^2 k^2}{2m} = \frac{L_x L_y}{4\pi\hbar^2}2m\,\frac{eB}{mc}\hbar\{n+1) - n\} = \frac{L_x L_y}{2\pi}\;\frac{eB}{\hbar c}} \tag{89}$$

There are two things worth noting: (i) Degeneracy $\boldsymbol{\mathcal{G}}$ is independent of $\boldsymbol{n}$, (ii) stronger is the magnetic field higher is the degeneracy.

The grand partition function can then be written by summing over all single particle states, as

$$\boldsymbol{\ln\mathbb{Z} = \ln\sum_{\epsilon}\ln\left[1 + e^{\beta(\mu-\epsilon(n,k_z)}\right] = \ln\sum_{n,p_z}\ln\left[1 + e^{\beta\left(\mu-\left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right)+\frac{\hbar^2 k_Z^2}{2m}\right)\right)}\right]} \tag{90}$$

$$\ln \mathbb{Z} = \mathcal{G} \int dz\, dk_z \sum_n \ln\left[1 + e^{\beta\left(\mu - \left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_Z^2}{2m}\right)\right)}\right]$$

$$= \mathcal{G}\frac{L_Z}{2\pi} \int dk_z \sum_n \ln\left[1 + e^{\beta\left(\mu - \left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_Z^2}{2m}\right)\right)}\right]$$

Or $$= \frac{L_x L_y}{(2\pi)^2}\frac{eB}{\hbar c} L_Z \int dk_z \sum_n \ln\left[1 + e^{\beta\left(\mu - \left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_Z^2}{2m}\right)\right)}\right] \qquad (91)$$

$$= \frac{V}{(2\pi)^2}\frac{m}{\hbar}\frac{eB}{mc} \int dk_z \sum_n \ln\left[1 + e^{\beta\left(\mu - \left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_Z^2}{2m}\right)\right)}\right]$$

$$= \frac{V}{(2\pi)^2}\frac{m\omega}{\hbar} \int dk_z \sum_n \ln\left[1 + e^{\beta\left(\mu - \left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_Z^2}{2m}\right)\right)}\right]$$

The integral obtained above can not be solved analytically in a closed form. However, in the high temperature limit, $e^{\beta\mu} \ll 1$, $\ln\left[1 + e^{\beta\left(\mu - \left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_Z^2}{2m}\right)\right)}\right]$ can be approximated using $\ln(1+x) \approx x$ so that

$$\ln \mathbb{Z} = \frac{V}{(2\pi)^2}\frac{m\omega}{\hbar} \int_{-\infty}^{\infty} dk_z \sum_n e^{\beta\left(\mu - \left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_Z^2}{2m}\right)\right)}$$

Or $$= \frac{V}{(2\pi)^2}\frac{m\omega}{\hbar} e^{\beta\mu} \int_{-\infty}^{\infty} dk_z\, e^{-\beta\left(\frac{\hbar^2 k_Z^2}{2m}\right)} \sum_n e^{-\beta\left(\frac{e\hbar B}{mc}\left(n+\frac{1}{2}\right)\right)} \qquad (92)$$

Noting that,

$$\int_{-\infty}^{\infty} dk_z\, e^{-\beta\left(\frac{\hbar^2 k_Z^2}{2m}\right)} = \sqrt{\frac{2\pi m k_B T}{\hbar^2}} = \frac{1}{\lambda} 2\pi$$

And

$$\sum_{n=0}^{\infty} e^{-\left(n+\frac{1}{2}\right)x} = \frac{e^{-\frac{x}{2}}}{1-e^{-x}} = \frac{1}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \frac{1}{2\sinh\left(\frac{x}{2}\right)}$$

Equation (92) becomes

Or
$$\ln \mathbb{Z} = \frac{V}{k_BT(2\pi)^2}\frac{mk_BT\omega}{\hbar}\, e^{\beta\mu}\sqrt{\frac{2\pi mk_BT}{\hbar^2}}\,\frac{1}{2\sinh\left(\beta\frac{\hbar\omega}{2}\right)} \tag{93}$$

Or
$$\ln \mathbb{Z} = \frac{Vh\omega}{k_BT(2\pi)^2}\frac{2\pi mk_BT}{h^2}\, e^{\beta\mu}\sqrt{\frac{2\pi mk_BT}{\hbar^2}}\,\frac{1}{2\sinh\left(\beta\frac{\hbar\omega}{2}\right)} \tag{94}$$

Or
$$\ln \mathbb{Z} = \frac{V}{k_BT}\frac{e\hbar B}{2mc}\, e^{\beta\mu}\,\frac{2\pi mk_BT}{h^2}\sqrt{\frac{2\pi mk_BT}{h^2}}\,\frac{1}{\sinh\left(\beta\frac{\hbar\omega}{2}\right)} \tag{95}$$

Or
$$\ln \mathbb{Z} = \frac{V}{\lambda^3}\, e^{\beta\mu}\,\frac{\frac{\mu_BB}{k_BT}}{\sinh(\beta\,\mu_BB\,)} = \frac{V}{\lambda^3}\,\frac{\mu_BB}{k_BT}\,\frac{e^{\beta\mu}}{\sinh(\beta\,\mu_BB\,)} \tag{96}$$

Putting $e^{\beta\mu} = Z$, and recalling that equilibrium number $\bar{N} = Z\frac{\partial \ln \mathbb{Z}}{\partial Z}$

We get

Or
$$\begin{aligned}\bar{N} = Z\left(\frac{\partial \ln \mathbb{Z}}{\partial Z}\right)_{B,V,T} &= V\frac{e\hbar}{2mC}\frac{B}{k_BT}\,\frac{2\pi mk_BT}{h^2}\sqrt{\frac{2\pi mk_BT}{h^2}}\,\frac{e^{\beta\mu}}{2\sinh\left(\beta\frac{\hbar\omega}{2}\right)}\\ &= \frac{V\mu_BB}{k_BT}\frac{1}{\lambda^3}\,\frac{e^{\beta\mu}}{2\sinh\left(\beta\frac{\hbar\omega}{2}\right)}\end{aligned} \tag{97}$$

Or
$$n = \frac{\bar{N}}{V} = \frac{\mu_BB}{k_BT}\frac{1}{\lambda^3}\,\frac{e^{\beta\mu}}{2\sinh\left(\beta\frac{\hbar\omega}{2}\right)} = e^{\beta\mu}\frac{1}{\lambda^3}\,\frac{\frac{\mu_BB}{k_BT}}{2\sinh\left(\frac{\mu_BB}{k_BT}\right)} \tag{98}$$

Furthermore, using (96) and (98), magnetic moment of the gas is given by

$$M = \frac{1}{\beta}\left(\frac{\partial}{\partial B}\ln \mathbb{Z}\right)_{Z,V,T} = \frac{V}{\lambda^3} e^{\beta\mu}\mu_B \frac{\partial}{\partial \frac{\mu B}{k_B T}}\left(\frac{\frac{\mu_B B}{k_B T}}{\sinh\left(\frac{\mu_B B}{k_B T}\right)}\right) =$$

Or

$$= \mu_B \left(\frac{V}{\lambda^3} \frac{\frac{\mu_B B}{k_B T}\ e^{\beta\mu}}{\sinh x}\right)\left(\frac{1}{\frac{\mu_B B}{k_B T}} - \coth\frac{\mu_B B}{k_B T}\right) \tag{99}$$

$$= \mu_B \bar{N} \left(\frac{1}{\frac{\mu_B B}{k_B T}} - \coth\frac{\mu_B B}{k_B T}\right)$$

Or

$$M = -\mu_B N \left(\coth x - \frac{1}{x}\right) = -N\mu_B L(x) \tag{100}$$

Where $= \frac{\mu_B B}{k_B T}$, $\bar{N} = N$ and $L(x)$ is the Langevin function we encountered in the study of paramagnetism.

However, note that for $x > 0$, $L(x) > 0$, therefore, the presence of negative sign, indicates that this is a signature of diamagnetism. For $x \ll 1$, the weak field case, $\coth x - \frac{1}{x} \sim \frac{x}{3}$, and hence

Or

$$M = -\mu_B N \frac{x}{3} = -\frac{1}{3} N\mu_B^2 \frac{B}{k_B T} \tag{101}$$

The corresponding diamagnetic susceptibility then can be written as

Or

$$\chi = \frac{M}{V} = -\mu_B \frac{N}{V}\frac{x}{3} = -\frac{1}{3} n\mu_B^2 \frac{B}{k_B T} \tag{102}$$

That is diamagnetic susceptibility is inversely proportional to temperature T, a law similar to Curie law.

7. Summary

In this module we have learnt

- That quantum statistical ensembles involves double averaging one corresponding to quantum mechanical averaging and the other corresponding to statistical averaging.
- That quantum effects become pronounced when the inter-particle distance between the particles is comparable to their average de Broglie Wavelength and wave functions of different particles overlap and system can-not be treated classically. The gas is then said to be degenerate. This happens when fermion gas has very low temperature or very high density.
- That Fermi distribution function in the limit $\frac{\epsilon_k-\mu}{k_BT} \gg 1$, transforms into $e^{-\frac{\epsilon_k-\mu}{k_BT}}$ which is nothing but Boltzmann distribution function.
- That at $T = 0$ Fermi distribution function is a step fumction such that for $\epsilon < \epsilon_F$ $f(\epsilon) = 1$ and for $\epsilon < \epsilon_F$ $f(\epsilon) = 0$.
- How to derive following physical quantities, for Fermi Gas at T=0

(a)	Number Density	$\frac{N}{V} = \frac{\mathcal{G}k_F^3}{3\,\pi^2}$
(b)	Fermi wave vector,	$k_F = \left(\frac{3\pi^2 N}{\mathcal{G}V}\right)^{\frac{1}{3}}$
(c)	Fermi Energy	$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{\mathcal{G}V}\right)^{\frac{2}{3}}$
(d)	Total Energy	$E_o = \frac{\mathcal{G}}{2}\frac{3N\epsilon_F}{5}$
(e)	Pressure	$P = \frac{\mathcal{G}}{2}\frac{2N}{5V}\epsilon_F$
(f)	Fermi Temperature	$T_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}}$

- How to derive following physical quantities, for Fermi Gas at $T \neq 0$ and for a non-degenerate fermi gas

	Physical Property	Fermi Gas at $T \neq 0$	Non-degenerate Fermi gas
(a)	Equation of State	$\frac{PV}{Nk_BT} = \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(Z)}$	$PV = Nk_BT$

(b)	Number density	$\frac{N}{V} = \frac{\mathcal{G}}{\lambda^3} f_{\frac{3}{2}}(Z)$	$\frac{N}{V} = \frac{\mathcal{G}}{\lambda^3}$
(c)	Internal Energy	$U = \frac{3}{2} N k_B T \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} = \frac{3}{2} PV$	$U = = \frac{3}{2} PV$
(d)	Specific heat	$\frac{C_V}{NK_B} = \frac{15}{4}\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \frac{9}{4}\left(\frac{f_{\frac{3}{2}}(Z)}{f_{\frac{1}{2}}(Z)}\right)$	$C_V = \frac{3}{2} N k_B$
(e)	Helmholtz free energy	$A = N k_B T \left(\ln Z - \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(Z)}\right)$	$A = N k_B T(\ln Z - 1)$
(f)	Entropy	$S = \left(\frac{5}{2} N k_B T \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)} - \ln Z\right)$	$S = = \left(\frac{5}{2} N k_B T - \ln\left(\frac{N\lambda^3}{V\mathcal{G}}\right)\right)$

- How to solve fermi integrals by Sommerfeld's approach and derive thermodynamic properties of Fermi Gas at finite but low temperature

	Physical Property	Fermi Gas at low and finite but low temperature
(a)	Equation of State	$\frac{PV}{N k_B T_F} = \frac{2}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \textit{higher order terms}\right]$
(b)	Number density	$\frac{N}{V} = = \frac{4\pi\mathcal{G}}{3}\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} (\mu)^{\frac{3}{2}}\left[1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2 + \textit{higher order terms}\right]$
(c)	Chemical Potential	$\mu = \left(\frac{3N}{4\pi\mathcal{G}V}\right)^{\frac{2}{3}} \frac{h^2}{2m}\left[1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2 + \textit{higher order terms}\right]^{-\frac{2}{3}}$ $\cong \epsilon_F[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \textit{higher order terms}]$
(d)	Internal Energy	$U = \frac{4\pi\mathcal{G}V}{5}\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \mu^{\frac{5}{2}}\left[1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2 + \textit{higher order terms}\right]$
(e)	Specific heat	$\frac{C_V}{N k_B} = \frac{\pi^2}{2}\frac{k_B T}{\epsilon_F} + \cdots = \frac{\pi^2}{2}\frac{T}{T_F} + \cdots$
(f)	Helmholtz free energy	$F = \frac{3}{5} N\epsilon_F\left[1 - \frac{5\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \textit{higher order terms}\right]$

(g)	Entropy	$S = \; Nk_{B_F}\left[\frac{\pi^2}{2}\frac{k_BT}{\epsilon_F} + \textit{higher order terms}\right]$

- That pauli paramagnetism is a result of contribution of spin alignment of electrons in the presence of an external magnetic field to the magnetic moment of a degenerate electron gas
- That Pauli Susceptibility at $T = 0$ is given by

$$\chi_0 = \frac{4\pi\mu_B^2\,(2m)^{\frac{3}{2}}}{h^3}(\epsilon_F)^{\frac{1}{2}}$$

- That Pauli Susceptibility at finite $T$is given by

$$\chi \approx \frac{3}{2}\frac{\mu_B^2\,N}{\epsilon_F V}\;\left[1+\;\frac{\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\right] = \chi_0\left[1+\;\frac{\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\right]$$

- That since in the presence of magnetic field a moving electron follows a helical path under the influence of Lorentz force, such that current produced by them induces a magnetic field in a direction opposite to external magnetic field resulting in diamagnetism called ***Landau diamagnetism***.
- That Landau diamagnetic susceptibility is given by

$$\chi_{dia} = \frac{M}{V} = \;-\mu_B\,N\left(\coth x - \frac{1}{x}\right) = -N\mu_B L(x)$$

Where $L(x)$ is Langevin function and $= \frac{\mu_B B}{k_BT}$, which for $x\; << 1$ i.e. for a weak field case gives

$$\chi_{dia} = \frac{M}{V} = \;-\mu_B\frac{N}{V}\frac{x}{3} = -\frac{1}{3}n\mu_B^2\frac{B}{k_BT}$$

Bibliography

1. Pathria R.K. and Beale P. D., “Statistical Mechanics”, 3rd Edition, Oxford: Elsiever Ltd., 2011
2. Brush S.G., “Statistical Physics and the Atomic Theory of Matter, from Boyle and Newton to Landau and Onsager”, New Jersey: Princeton University Press, 1983.
3. Landau L.D., Lifshitz E.M., Statistical Physics, vol 5, part1, 3rd Edition, Oxford:Pergmon Press, 1982.
4. Panat P.V., “Thermodynamics and Statistical Mechanics”, New Delhi: Narosa Publishing House Pvt. Ltd., 2008
5. Pal B.P., “ An Introductory Course of Statistical Mechanics, New Delhi: Narosa Publishing House Pvt. Ltd., 2008
6. Kubo R., “Statistical Mechanics: An Advanced Course with problems and Solutions,” Amsterdam: North Holland Publishing Company, 1965.

Appendices

A1 Calculation of Density of States

In module VII, we looked at the problem of particle in a cubical box and found solution of a particle in a cubical box, without loss of generality we can consider this particle in a cuboid of length L_x, L_y *and* L_z. The solution of the schrodinger equation in three dimensions can be written as

$$\psi = A \sin xk_x \sin yk_y \sin zk_z$$

Where $k_x = \frac{n_x \pi}{L_x}$, $k_y = \frac{n_y \pi}{L_y}$ and $k_z = \frac{n_z \pi}{L_z}$ are the components of the wave vector $\vec{k}$; n_x, n_y and n_z are three quantum numbers taking integer values 1,2,3 …. ………. and $A = \left(\frac{2}{L_x L_y L_z}\right)^{3/2}$. Hence momentum p is quantized and energy of the energy levels is

$$E = \frac{\hbar^2 \pi^2}{2m}\left[\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right]$$

We note that as size of the box becomes larger and larger, energy becomes almost continuous. In the limiting case, $E_{n_x,n_y,n_z} - E_{n_x-1,n_y-1,n_z-1} = \Delta E_{n_x,n_y,n_z}$ become very small. So E can be treated as a continuous variable of quantum numbers. For a fixed value of k_y and k_z, the number Δn_x of possible integers when k_x lies between k_x and $k_x + dk_x$ equals

$$\Delta n_x = \frac{L_x}{\pi} dk_x$$

Therefore, the number of states between k_x and $k_x + dk_x$, between k_y and $k_y + dk_y$, **between** k_z **and** $k_z + dk_z$ are

$$\Delta n_x \Delta n_y \Delta n_z = \frac{L_x}{\pi}\frac{L_y}{\pi}\frac{L_z}{\pi} dk_x dk_y dk_z = \frac{V}{\pi^3}\, d^3k$$

If we suppose $\mathcal{D}(k)$ is the density of states i.e. number of states in the unit interval around k. Then the number of states between k and $k + dk$ is $\mathcal{D}(k)\, \mathrm{d}^3k = \frac{V}{8\pi^3}\, d^3k$, here the factor of 8 in the denominator arises from the fact that since k_x, k_y and k_z are positive definite only volume of one octant should contribute.

Thus

$$\mathcal{D}(k) = \frac{V}{(2\pi)^3}$$

Generalising it to one-dimension, two dimensions, three dimensions and d dimensions we get

$$\sum_k f(\epsilon_k) = \begin{cases} \frac{L}{(2\pi)}\int f(\epsilon_k)\,dk \;=\; \int \mathcal{D}(k) f(\epsilon_k) d^1k \;\; in\ one\ dimension \\ \frac{A}{(2\pi)^2}\int f(\epsilon_k) d^2k \;=\; \int \mathcal{D}(k) f(\epsilon_k) d^2k \;\; in\ two\ dimensions \\ \frac{V}{(2\pi)^3}\int f(\epsilon_k) d^3k \;=\; \int \mathcal{D}(k) f(\epsilon_k) d^3k \;\; in\ three\ dimensions \\ \frac{L^d}{(2\pi)^d}\int f(\epsilon_k) d^dk \;=\; \int \mathcal{D}(k) f(\epsilon_k) d^dk \;\; in\ \ d\ dimensions \end{cases}$$

A2 Fermi Dirac Integrals

The integrals of the following type encountered in this module are called Fermi Dirac integrals

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{(Z^{-1}e^x + 1)} dx$$

It has following interesting properties:

(a) For small Z, the factor $\frac{1}{(Z^{-1}e^x+1)}$ in the integrand can be written as $Ze^{-x}(1 + Ze^{-x})^{-1}$ and can be expanded as given below

$$Ze^{-x}(1 + Ze^{-x})^{-1} = Ze^{-x}(1 + (-1)Ze^{-x} + \frac{(-1)(-2)}{2!}(Ze^{-x})^2 + \frac{(-1)(-2)(-3)}{3!}(Ze^{-x})^3 + \cdots$$

$$= (Ze^{-x} + (-1)(Ze^{-x})^2 + (Ze^{-x})^3 + (-1)(Ze^{-x})^4 + \cdots = \sum_{l=1}^{\infty} (-1)^{l-1}\, Ze^{-x}$$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty x^{\nu-1} \sum_{l=1}^{\infty} (-1)^{l-1}\, (Ze^{-x})^l\, dx$$

Let us now look at the lth term of the integral

$$(-1)^{l-1} Z^l \int_0^\infty x^{\nu-1} e^{-lx} \;\; dx$$

Put $lx = y$ in the above integral, we get $dx = \frac{dy}{l}$ and it takes the form

$$(-1)^{l-1}\frac{Z^l}{l^\nu}\int_0^\infty (y)^{\nu-1}\, e^{-y}\; dy = (-1)^{l-1}\frac{Z^l}{l^\nu}\,\Gamma(\nu)$$

Therefore,

$$f_\nu(z) = \sum_{l=1}^{\infty}(-1)^{l-1}\frac{Z^l}{l^\nu} = z - \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} - \cdots$$

(b) The integral $f_\nu(z)$ satisfies the following recurrence relation which can be easily verified from the series given above

$$Z\frac{\partial f_\nu(z)}{\partial z} = f_{\nu-1}(z)$$